 Sample question solutions

Proving simple results involving numbers

1. If is odd then is odd.

If is an odd integer, then for some integer (definition of an odd)

 is odd as is an integer

 is odd

1. Prove that if and are both odd, then is odd; otherwise is even. (NESA exemplar question)

There are three possible combinations of and

* Both are odd, let and

 is odd as and are integers

if and are both odd, then is odd.

* Both are even, let and

is even as and are integers

if and are both even, then is even.

* One is odd and one is even, let and

 is even as and are integers

if one is even and one is odd, then is even.

 if and are both odd, then is odd; otherwise is even.

1. The product of two consecutive even counting numbers is a multiple of .

If is an even number then and are two consecutive even numbers and is the product of two consecutive even numbers.

 for some integer (definition of an odd)

.

 is a multiple of 4 as is an integer.

 The product of two consecutive even counting numbers is a multiple of .

Proof by contradiction

1. For all integers , if  is odd, then is odd.

Assume there exists an integer , if is odd, then is even

If this case:

 and where and are integers (definition of odd and even)

But this is impossible. is an integer as and are integers

Therefore our assumption that there exists an integer such that if is odd, then is even

is false, so must be odd.

1. Every factor of an odd number is odd.

Let be an odd number and assume there exists an even factor, , of .

, where and are integers.

Let (definition of an even number)

 is even as and are integers. So is also even, which is a contradiction. Therefore is odd.

every factor of an odd number cannot be even and thus is odd.

1. The negative of any irrational number is irrational.

Assume there exists an irrational number , such that its negative is rational.

The negative of is rational, for some integers and (

Multiply by −1

 and are integers (since is an integer)

 by the definition of a rational number is rational, which is a contradiction.

There does not exists an irrational number , such that its negative is rational, the negative of any irrational number is irrational.

1. For all integers , if  is even, then is even.

Assume there exists an integer , if is even, then is odd

If this case:

 and where and are integers (definition of odd and even)

But this is impossible. is an integer as and are integers.

Therefore our assumption that there exists an integer such that if is even, then is odd

is false, so must be even.

1. Prove that if is a positive integer then is always irrational.

Assume there exist a positive integer n such that is rational, then:

 where and are integers with no common factors (except 1).

Squaring both sides:

, hence is even and is even. Let , where is an integer (definition of even).

, is odd so is even and is even.

So and are both even and are therefore divisible by 2, i.e. have a common factor.

This is a contradiction to our assumption that they have no common factors.

Hence is irrational.

1. There is no greatest even integer.

Assume there is a greatest even integer . Then

 for every integer

If we add one to an integer we obtain another integer

 is an integer.

If there were a greatest integer, we could add 1 to it to obtain an integer that is greater.

This is a contradiction, no greatest integer can exist.

1. Prove that in any group of  people, there are at least two who are acquainted with the same number of people.

Assume there are no people who are acquainted with the same number of people and therefore there must be at least different possibilities for the number of acquaintances.

In a group of people, a person may have acquaintances.

If someone has acquaintances at the party, then no one at the party has acquaintances at the party, and if someone has friends at the party, then no one has 0 acquaintances at the party. Hence the number of acquaintances a person may have is or This is possibilities for the number of acquaintances. This is a contradiction and therefore there are at least two who are acquainted with the same number of people.

1. Prove is irrational.

Assume is rational.

Then

Where and are integers with no common factors,

Squaring both sides:

 is even and is also even. Let , where is an integer (definition of even)

 is even and is also even.

So and are both even and are divisible by 2, i.e. have a common factor.

This is a contradiction to our assumption that they have no common factors.

 is rational is false. is irrational.

.

1. Prove that there exist no integers and for which .

Assume integers and can be found for which .

Divide both sides by 6:

 .

This is a contradiction, as is an integer but is not.

Therefore, no integers and exist for which .

Using examples

1. Prove there exists a number which is half the sum of its positive factors.

Consider the number 6.

The factors of 6 are 1,2,3,6.

The number 6 is half the sum of its positive factors.

 there exists a number which is half the sum of its positive factors.

1. Prove a prime number , such that 2 and are also prime numbers

Consider the prime number 5.

which is a prime number

 which as a prime number

 there exists a prime number , such that 2 and are also prime numbers

Note: This also works for other prime numbers such as 11 and 17

1. Prove a function , such that

Consider the function

 there exists a function , such that

Note: This is also true for all functions of the form where is a real number.

Using counter-examples

* 1. All prime numbers are odd (NESA topic guidance)

Consider the number 2.

2 is a prime number and 2 is even

 all prime numbers are not odd and the original statement to be false.

* 1. then

Choose any integer for and then choose

Let and

This is an example where but , disproving the original statement.

* 1. A quadrilateral with four equal sides is a square

Draw a rhombus where no angles are .

This is a quadrilateral with for equal sides which is not a square, disproving the original statement.

Consider

 there exists an , , disproving the original statement.

Consider a value for such that

Consider

 there exists an , , disproving the original statement.

Proofs involving inequalities

1. Prove that if are real and not all equal then . Deduce that if additionally , then . (NESA topic guidance)
* Part 1: Prove that if are real and not all equal then

, and

but are real and not all equal, i.e. andand

 (dividing by 2)

* Part 2: If , then

From part 1:

 (dividing by 3)

1. Prove and hence
* Part 1: Prove
* Part 2: Prove

From part 1:

(from part 1)

1. If  are positive real numbers and , prove that  (2011 HSC)

Consider the numerator only as the denominator is positive as and are all positive.

as  are positive real numbers.

The numerator and denominator are both greater than zero,

1. Prove .

The left hand side appears to come from

The triangle inequality

1. Prove reverse triangle inequality: Prove that if and are real numbers, then

Triangle inequality:

Let and

Let and

Consider

If

If

 or

1. Prove

, from the triangle inequality:

, but

The arithmetic and geometric mean

1. Extension 2 HSC 2012
2. Prove that where and

Method 1:

Method 2:

1. If , show that

 (multiply by . Note: since

1. Let and be positive integers with , prove that

From b):

, let and .

 as

We have shown the left hand side of the inequality, we now need to show its maximum value is

From a):

, considering that we want let and

Method 2: Consider the maximum value of (note: this is a longer method)

Let

Solve

 is a local maximum.

1. For integers, prove that

Using the result from part c) and letting take the values

1. Given that , prove that, if , , then and (NESA topic guidance)
* Part 1: Given that , prove that, if , , then

Use the relationship between the arithmetic mean and geometric mean for two non-negative numbers:

* Part 2: Given that , prove that, if , , then

Use the relationship between the arithmetic mean and geometric mean for two non-negative numbers:

1. If the product of two positive numbers is 64, what is the minimum value of their sum?

Let the numbers beand, then

The minimum value of the sum is 16

1. If and , then find the minimum value of
2. Find the minimum value of for .

Use the relationship between the arithmetic mean and geometric mean for two non-negative numbers:

1. Show that the rectangle of the largest possible area, for a given perimeter, is a square.

Let the rectangle have dimensions and

Rearranging:

Use the relationship between the arithmetic mean and geometric mean for two non-negative numbers:

The maximum area is

Therefore the maximum area is when the rectangle is a square.

1. For positive reals numbers and z, prove

By our theorem, or

Similarly and

1. A jelly shop sells its products in two different sets: 3 red jelly cubes and 3 green jelly rectangular prisms. The three red cubes are of side lengths and where , while the three green rectangular prisms are identical with dimensions . Which option would give you more jelly?

Volume of the red cubes

Volume of the green cubes

According to the arithmetic mean – geometric mean inequality for three terms, and :

The inequality is only equal if so the volumes can never be equal as

Therefore the volume of the red cubes will be greater.

Prove further inequalities

1. Prove . Hence deduce that

Use the relationship between the arithmetic mean and geometric mean for two non-negative numbers and multiply by two

1.

see questions 7 from The arithmetic and geometric mean.

Adding the left and right

1. Prove . Hence deduce that:

Use the relationship between the arithmetic mean and geometric mean for two non-negative numbers:

 (simplify both sides)

1. NESA Example: Let be a fixed, non-zero number satisfying .
2. Use the method of mathematical induction to prove that for

**Step 1: Base case:** Prove for

 as

**Step 2:** Assume that the statement is true for

i.e.

**Step 3: Inductive step:** Prove the statement to be true for .

 as

**Step 4: Concluding statement**

1. Deduce that for
2. Extension 2 HSC 2007
3. Show that for

Let

For is a decreasing function and

for

1. Let . Show that the graph of is concave up for

Show that for

 since then or

 is concave up for

1. By considering the first two derivatives of , show that for

Show or for

We need to show the function is always positive when

Since and the curve has a minimum turning point at and the curve is increasing in value when Since when as it is increasing in value.

 for