 Square root of a complex number

Sample methods and examples

Question 1

Find the square roots of $-3+4i$

Solution 1

Let $z=\sqrt{p+qi}$

$$a+bi=\sqrt{-3+4i}$$

$$\left(a+bi\right)^{2}=-3+4i$$

$$a^{2}+2abi-b^{2}=-3+4i$$

$$a^{2}-b^{2}+2abi=-3+4i$$

Equate the coefficients:

$$\left(1\right) a^{2}-b^{2}=-3$$

$$\left(2\right) 2ab=4$$

$b=\frac{4}{2a}=\frac{2}{a}$ and $a=\frac{4}{2b}=\frac{2}{b}$

Substitute $b=\frac{2}{a}$ into (1)

$$a^{2}-\left(\frac{2}{a}\right)^{2}=-3$$

$$a^{2}-\frac{4}{a^{2}}=-3$$

$$a^{4}-4=-3a^{2}$$

$$a^{4}+3a^{2}-4=0$$

Using the quadratic formula:

$$a^{2}=\frac{-\left(3\right)\pm \sqrt{\left(3\right)^{2}-4×1×\left(-4\right)}}{2×1}$$

$$a^{2}=\frac{-3\pm \sqrt{9+16}}{2}$$

$$a^{2}=\frac{-3\pm \sqrt{25}}{2}$$

$$a^{2}=\frac{-3\pm 5}{2}$$

$$a=\pm \sqrt{\frac{-3\pm 5}{2}}$$

$a=\pm \sqrt{\frac{-3+5}{2}}$, since $a$ is real

$$a=\pm \sqrt{\frac{2}{2}}$$

$$a=\pm 1 $$

$$b=\frac{2}{a}$$

When $a=1, b=2$

When $a=-1, b=-2$

The square roots of $-3+4i$ are $1+2i$ and $-1-2i$

Two methods to check a solution:

1. Square the roots to check they equal the original complex number **i.e.** Show:
$\left(1+2i\right)^{2}=\left(1+2i\right)\left(1+2i\right)=-3+4i$ and $\left(-1-2i\right)^{2}=\left(-1-2i\right)\left(-1-2i\right)=-3+4i$
2. Check by graphing the two simultaneous equations using graphing software.

For $a^{2}-b^{2}=-3$, graph$ y^{2}-x^{2}=-3 $and for $2ab=4$, graph$ 2xy=4$ and then read the points of intersection. So, $x=b=2$ when $y=a=1$ and $x=b=-2$ when $y=a=-1$

Question 2

Find the square root of $p+qi$ leaving the answer in the form $z=a+bi$

Solution 2

Let $z=\sqrt{p+qi}$

$$a+bi=\sqrt{p+qi}$$

$$\left(a+bi\right)^{2}=p+qi$$

$$a^{2}+2abi-b^{2}=p+qi$$

$$a^{2}-b^{2}+2abi=p+qi$$

Equating the coefficients:

$$\left(1\right) a^{2}-b^{2}=p$$

$$\left(2\right) 2ab=q$$

$b=\frac{q}{2a}$ and $a=\frac{q}{2b}$

Substitute $b=\frac{q}{2a}$ into (1)

$$a^{2}-\left(\frac{q}{2a}\right)^{2}=p$$

$$a^{2}-\frac{q^{2}}{4a^{2}}=p$$

$$4a^{4}-q^{2}=4pa^{2}$$

$$4a^{4}-4pa^{2}-q^{2}=0$$

Using the quadratic formula:

$$a^{2}=\frac{-\left(-4p\right)\pm \sqrt{\left(-4p\right)^{2}-4×4×\left(-q^{2}\right)}}{2×4}$$

$$a^{2}=\frac{4p\pm \sqrt{16p^{2}+16q^{2}}}{8}$$

$$a^{2}=\frac{4p\pm 4\sqrt{p^{2}+q^{2}}}{8}$$

$$a^{2}=\frac{p\pm \sqrt{p^{2}+q^{2}}}{2}$$

$a^{2}=\frac{Re(z)\pm |z|}{2}$, given the modulus of $z$ **i.e.** $\left|z\right|=\sqrt{p^{2}+q^{2}}$

$$a=\pm \sqrt{\frac{Re(z)\pm |z|}{2}}$$

$a=\pm \sqrt{\frac{Re\left(z\right)+|z|}{2}}$, since $a$ is real

$b=\frac{q}{2a}$ or $b=\frac{Im(z)}{2a}$