 Year 11 mathematics extension 1

| ME-A1 Working with combinatorics | Unit duration |
| --- | --- |
| The topic Combinatorics involves counting and ordering as well as exploring arrangements, patterns, symmetry and other methods to generalise and predict outcomes. The consideration of the expansion of , where 𝑛 is a positive integer, draws together aspects of number theory and probability theory. A knowledge of combinatorics is useful when considering situations and solving problems involving counting, sorting and arranging. Efficient counting methods have many applications and are used in the study of probability. The study of combinatorics is important in developing students’ ability to generalise situations, to explore patterns and to ensure the consideration of all outcomes in situations such as the placement of people or objects, setting-up of surveys, jury or committee selection and design. | 2-3 weeks |

| Subtopic focus | Outcomes |
| --- | --- |
| The principal focus of this subtopic is to develop students’ understanding and proficiency with permutations and combinations and their relevance to the binomial coefficients. Students develop proficiency in ordering and counting techniques in both restricted and unrestricted situations. The binomial expansion is introduced, Pascal’s triangle is constructed and related identities are proved. The material studied provides the basis for more advanced work, where the binomial expansion is extended to cases for rational values of 𝑛, and applications in calculus are explored. | A student:   * uses concepts of permutations and combinations to solve problems involving counting or ordering ME11-5 * uses appropriate technology to investigate, organise and interpret information to solve problems in a range of contexts ME11-6 * communicates making comprehensive use of mathematical language, notation, diagrams and graphs ME11-7 |

| Prerequisite knowledge | Assessment strategies |
| --- | --- |
| The material in this topic builds on content from MA-S1 S1.1. Students should be capable of drawing probability trees and listing possible outcomes; able to find the probability of an event and use complementary events; understand the multiplication and addition principles; and describe events using language of 'at least', exclusive 'or' (A or B but not both), inclusive 'or' (A or B or both) and 'and'. | **Summative Assessment:** What are the chances of winning the lottery? Students investigate the probability of winning various lotteries. They calculate the theoretical probability and use computer simulations to calculate experimental probability values to make a judgment about which lotteries offer the best and least chances of winning. Students could present their findings to others to promote understandings about gambling. |

All outcomes referred to in this unit come from [Mathematics Extension 1](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-extension-1-2017) Syllabus  
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Glossary of terms

| Term | Description |
| --- | --- |
| Arranging n objects in an ordered list | The number of ways to arrange different objects in an ordered list is given by |
| combination  | A combination is a selection of distinct objects from distinct objects, where order is not important. The number of such combinations is denoted by or , and is given by: |
| factorial | The product of the first positive integers is called the factorial of and is denoted by .  By definition: |
| fundamental counting principle | The fundamental counting principle states that if one event has possible outcomes and a second independent event has possible outcomes, then there are a total of possible outcomes for the two combined events. |
| permutation  | A permutation is an arrangement of distinct objects taken from distinct objects where order is important.  The number of such permutations is denoted by and is equal to:  The number of permutations of objects is . |

| Background knowledge |
| --- |
| ‘Combinatorial thinking involves investigating some cases, [asking] how am I sure that I have counted all the cases, systematically generating all cases, changing the problem into another combinatorial problem and understanding combinatorial reasoning’.  **Resource:** [What do I mean by combinatorial thinking?](https://www.researchgate.net/publication/251713394_What_do_I_mean_by_combinatorial_thinking) |

Lesson sequence

| Lesson sequence | Content  Students learn to: | Suggested teaching strategies and resources | Date and initial | Comments, feedback, additional resources used |
| --- | --- | --- | --- | --- |
| Introduction to the fundamental counting principle  (2 lessons) | **A1.1: Permutations and combinations**   * list and count the number of ways an event can occur * use the fundamental counting principle (also known as the multiplication principle) | **Introduction to the fundamental counting principle**   * Students develop a need for the fundamental counting principle by investigating the number of [license plates](https://docs.google.com/document/d/1pur8yjMLGEKXXkqCYYyFkybEY_-918dZslmWNSPXaKg/edit?usp=sharing). * The fundamental counting principle: Suppose a choice is to be made in two stages. If there are a choices for the first stage and b choices for the second stage, no matter what choice has been made at the first stage, then there are choices altogether. If the choice is to be made in stages and if for each , there are choices for the th stage then there are choices altogether (from [Australian Curriculum](https://www.australiancurriculum.edu.au/media/1191/specialist-mathematics-glossary.pdf))   **Suggested examples**   * Menus at a restaurant. e.g. If there are 3 different entrees, 5 different main meals and 4 different desserts. In how many ways can I choose an entree, main and dessert? * Outfits (clothing) * Forming numbers from pre-specified digits * PINS (no repetition) * Routes between cities * Students are directed to explore a Subway restaurant store menu by physically going to a store (in their own time) or by accessing it through its website to calculate how many different sandwiches can be created. Each sandwich consists of a type of bread, meat filling, cheese filling, exactly 5 salad items and a sauce. Reflection questions could revolve around the effect of having optional items or a range of salad items (1 to 5) on the calculation. * This activity can be done with other food retail outlets such as Domino’s Pizza. * Possible extension activities could include:   + [Password strength](http://blog.recursiveprocess.com/2011/08/30/password-strength/)   + [Safe cracking](https://thescamdog.wordpress.com/2012/02/24/safe-cracking-in-3-acts/) |  |  |
| Representing arrangements in a line using factorial notation  (1 lesson) | * use factorial notation to describe and determine the number of ways different items can be arranged in a line or a circle | **Introducing factorial notation**   * Students develop a need for factorial notation. Students work in pairs or small groups and investigate/count the number of ways people can be arranged in a line for a photograph. Different scenarios should be explored, for example:   + the number of arrangements of 3 people   + the number of arrangements of 4 people   + the number of arrangements of 3 people when there are 4 people to choose from   **Calculating the number of arrangements in a line**   * Students are encouraged to develop generalisations to predict the number of arrangements of people in a line. * Students should consider the question ‘How do I know I have listed and counted all the possibilities?’ This question encourages students to make an organised list or construct a tree diagram or table to ensure that they have indeed exhausted all possibilities. Resource: [Photographs and committees: Activities that help students discover permutations and combinations](https://drive.google.com/a/education.nsw.gov.au/file/d/1hII1HF1GndMnsbaxXBiuXEPtsXthgt-K/view?usp=sharing) * Reflection questions should ideally lead to at least a basic understanding of factorials and the letterbox (space filling) technique for counting arrangements which can be developed in future lessons. A detailed explanation of the reason for each numerical entry into a space should be modelled to students. * Resource: This [Introductions to Combinatorics](https://www.khanacademy.org/partner-content/pixar/crowds/crowds-1/a/start-here-crowds) uses robots from Pixar movies. Using this resource students can count, for example, [the number of 3 colour arrangements from 4 colours](https://www.khanacademy.org/partner-content/pixar/crowds/crowds2/p/building-snake-bots). * The number of ways to arrange n different objects in an ordered list is given by n factorial, for example, ([Australian curriculum glossary](https://www.australiancurriculum.edu.au/media/1191/specialist-mathematics-glossary.pdf))   **Examples:**   * Arranging distinct letters/numbers.(no repetition)   + How many three letter ‘words’ can be made from the letters A, B and C if each letter may be used only once?   + How many different four digit numbers can be made from the digits 1, 2, 3, 4 if each number may be used only once?   + How many different ‘words’ can be made from the letters in the word ‘question’?   + How many different 5 letter ‘words’ can be made from the letters in the word ‘question’? * Lines of people * Books on a shelf |  |  |
| Representing arrangements in a circle using factorial notation  (1 lesson) | * use factorial notation to describe and determine the number of ways n different items can be arranged in a line or a circle | **Calculating the number of arrangements in a circle**   * Students develop a need for a different rule for arrangements in a circle by creating an arrangement of a small number of students (4 or 5) around a table and investigate/count the number of arrangements by physically moving students. Guide the discussion to identify arrangements that are not distinct, for example, arrangements obtained by rotating each student a fixed number of places. Ask students for strategies that address this situation. Ideally, with some guidance, they should suggest fixing one of the students and arranging the remaining students around that student. Connect this idea to the formula of * The number of ways n things can be arranged in a circle with no restrictions is   **Example**   * In how many ways can 8 people be seated around a circular table?   **Calculating the number of arrangements for beads on a circular wire or a bracelet**   * Students see a need for a different rule for arrangements of beads on a circular wire. * The number of ways n beads can be arranged on a bracelet is . As a necklace can be turned over, clockwise and anti-clockwise arrangements are the same. Resource – [Beads on a wire](https://nrich.maths.org/2048)   **Example**   * In how many ways can 6 beads be arranged on a bracelet?   **Zero factorial**   * Explore [Why is zero factorial equal to one?](https://www.youtube.com/watch?v=Mfk_L4Nx2ZI) (duration 7:31) |  |  |
| Arrangements with repeating items  (1 lesson) | * use factorial notation to describe and determine the number of ways n different items can be arranged in a line or a circle   + solve problems involving cases where some items are not distinct (excluding arrangements in a circle) | **Calculating the number of arrangements with repeating or identical items**   * Students to list the arrangements of the letters in a word where there is one pair of repeated letters for example, ‘speed’. Choose a word that does not contain too many letters. Use two different colours for the repeated letters to make it easy to distinguish them from each other. Ask students if arrangements with the repeated letters interchanged are different. Count the number of distinct arrangements. In pairs or small groups, ask students to repeat this activity with a word that contains 3 and 4 repeated letters, for example, ‘tattle’ and ‘possess’. The aim is for students to discover the relationship and formula. * The number of permutations of n objects of which are identical, another are identical, are identical, and so on   Examples   * How many different ‘words’ can be formed from the letters in the word ‘different’? * How many different 4 digit PINs can be formed from the numbers 1, 2, 3, 4 if repetition of digits is allowed? * A and B sitting together/apart, boys/girls |  |  |
| Introduction to permutations  (1 lesson) | * understand and use permutations to solve problems (ACMSM001)   + understand and use the notation and the formula | Using to calculate the number of permutations   * Students develop a need for a rule for counting permutations. * Following on from the earlier photographs activity students consider the following scenario: * Suppose you have a group of people but you only want of them at a time to be in your photograph. How many different photographs can you make? * Resource: [Photographs and committees: Activities that help students discover permutations and combinations](https://drive.google.com/a/education.nsw.gov.au/file/d/1hII1HF1GndMnsbaxXBiuXEPtsXthgt-K/view?usp=sharing) * Students should use their knowledge of factorial notation to come up with a rule for counting arrangements of things taking at a time. * A permutation of n objects is an arrangement or rearrangement of n objects where order is important. The number of arrangements of n different objects is n! The number of permutations of n objects taken r at a time is denoted by and is equal to * Students are to complete the activity, “[The Door Lock](http://blog.mrmeyer.com/2009/my-lesson-plan-the-door-lock/)” Students should explore, ‘What characteristics make for a good lock?’ * Students could also consider [information leakage](https://www.schneier.com/blog/archives/2009/07/information_lea_1.html).   Examples   * How many ways can a captain, a vice captain and a senior prefect be chosen from a group of 10 students? * Find the number of ways of sitting at a rectangular table with 4 seats on each side if one side faces a painting and the other does not. |  |  |
| Solving problems involving permutations with restrictions  (2 lessons) | * solve problems involving permutations and restrictions with or without repeated objects (ACMSM004) | Arrangements of distinct/unlike objects in a line  Examples   * Find the number of ways of positioning 7 students in a straight line if Francis and Ramzi are to fill the end positions? * How many ways can 4 males and 4 females be arranged in a straight line if they must alternate? * In how many ways can 8 boys sit in an 8-oared boat if 3 of the crew can only row on the stroke side and one particular boy must be on the bow side? * A five letter ‘word’ is to be made from the letters of the word CAMBRIDGE. How many ways can this be done if:   + The first letter has to be C?   + The last letter has to be E?   + There are no vowels?   + All the vowels must be included?   Arrangements of unlike elements in a line and involving special conditions  Example   * Find the number of ways of positioning A, B, C, D,E, F in 6 seats in line if A and B are to fill the end positions   **Arrangements involving groups**  Example   * Find the number of ways of positioning A, B, C, D, E and F if A, B and C are to be together.   **Arrangements involving the use of complementary events in space filling**  Example   * Find the number of ways of positioning A, B, C, D, E, F in 6 seats in line if A and B are together but E and F are not.   **Arrangements with repetitions**   * Students should be led by carefully considering suitable examples to a general method for arranging n objects of which are of one type, are of a second type and r are of a third type.   **Examples**   * How many ways can the letters of the word ‘exceed’ be arranged? * 3 green, 4 blue and 5 red cards are placed alongside one another. Find the number of colour patterns that can be formed.   **Arrangements of objects some of which are to be separated**  Example   * Find the number of 7 letter words which can be formed from the letters of the word ‘display’ if no 2 vowels are to be together. * **Note:** Decide on the number of ways of arranging the consonants D S P L Y. There are such arrangements. There are 6 possible places for the first vowel and 5 places for the second vowel. There are 3600 such ‘words’.   **Arrangements in a circle**   * Determine the number of arrangements of unlike elements in a circle, arrangements involving special conditions and arrangements involving groups.   Example   * A group of 10 people sit around a table. In how many ways can they be arranged:   + with no restrictions?   + if 2 particular people want to sit together?   + if 2 particular people cannot sit together?   + if 3 particular people want to sit together?   + if two particular people must sit opposite each other? |  |  |
| Introduction to combinations  (2 lessons) | * understand and use combinations to solve problems (ACMSM007)   + understand and use the notations and and the formula  (ACMMM045, ACMSM008) | **Investigating the idea of combinations**   * Selection of objects from a group of objects should developed by realising that order of choice is irrelevant and that the objects can be considered to be identical. * Class activity (Committees): The result for should initially be developed through a series of exercises of the following type:   + How many different committees of various sizes can be formed from a group of four members if the committee is of sizes 4, 3, 2, 1, 0?   + If another person joins the group, how many committees can be formed of sizes 5, 4, 3, 2, 1, 0?   + Students are then given an opportunity to make a conjecture about the rule for the formula for combinations (committees) based on their knowledge of permutations (photographs) and the formula for.   + Students’ understanding should then be extended to exercises involving larger numbers and restrictions as in the examples below. * This [Counting graphic](http://students.brown.edu/seeing-theory/compound-probability/index.html) illustrates the difference between permutations and combinations using a tree diagram.   Examples   * How many different 3 person committees can be chosen from 30 people? * A committee of 6 people is to be selected randomly from a group of 11 men and 12 women. Find the number of possible committees if:   + There is no restriction as to who is on the committee   + There are to be 3 men and 3 women   + A particular man is to be included   + A particular woman is not included   **Selection of groups of identical sizes**  Example   * 22 students are divided into 2 hockey teams each containing 11 players? How many team combinations are possible?   Note: This result is |  |  |
| Solve practical problems involving permutations and combinations  (2 lessons) | * solve practical problems involving permutations and combinations, including those involving simple probability situations AAM Critical and creative thinking icon | **Solving problems in a variety of contexts**   * Students are now given the opportunity to apply all these techniques to problems involving both permutations and combinations as well as to those involving probability. Special attention should be given during this practice to choosing the most appropriate technique for the solution of a problem. Students should give a detailed explanation of their thinking in each case. This may be in written or diagrammatic form.   **Note**: for problems involving probability the number of arrangements corresponding to the particular event and to the sample space should be found separately.  Examples   * If 4 mathematics books are selected from 6 different mathematics books and 3 English books are chosen from 5 different English books, how many ways can the seven books be arranged on a shelf if:   + There are no restrictions?   + The 4 mathematics books remain together?   + A mathematics book is at the beginning of the shelf?   + mathematics and English books alternate?   + mathematics is at the beginning and an English book is in the middle of the shelf? * There are 10 red, 10 green, 10 blue and 10 yellow cards in a pack of 40 cards.   + Find the probability of selecting 10 cards that are the same colour.   + Find the probability that at least 1 card is red. (complementary events)   + Find the probability that at least 8 cards are red. (mutually exclusive events)   + Find the probability of obtaining exactly 5 red cards if it is known that at least 1 card was red. (conditional probability)   **HSC example** (From 2007 Extension 1 Question 5b)   * Mr and Mrs Roberts and their four children go to the theatre. They are randomly allocated six adjacent seats in a single row. What is the probability that the four children are allocated seats next to each other? * Mr and Mrs Roberts and their four children are to be seated around a circular table at a restaurant. If their seats are to be allocated randomly, what is the probability that the four children sit next to each other?   **Further applications and modelling**   * Further [application areas of combinatorics](https://pdfs.semanticscholar.org/ffc6/601b91268f6bbfa7426055b38f91eb2d11ea.pdf) could be explored including the [application of permutations and combinations in telecommunications.](https://www.researchgate.net/publication/50315356_APPLICATION_OF_THE_PRINCIPLES_OF_PERMUTATION_AND_COMBINATION_IN_MATHEMATICS_IN_TELECOMMUNICATIONS) |  |  |
| Pigeonhole principle  (1 – 2 lessons) | * solve simple problems and prove results using the pigeonhole principle (ACMSM006)   + understand that if there are 𝑛 pigeonholes and 𝑛 + 1 pigeons to go into them, then at least one pigeonhole must hold 2 or more pigeons   + generalise to: If 𝑛 pigeons are sitting in 𝑘 pigeonholes, where 𝑛 > 𝑘, then there is at least one pigeonhole with at least pigeons in it   + prove the pigeonhole principle | **Introduction to pigeonhole principle**   * The pigeonhole principle finds its usefulness in many areas including geometry, number theory, graph theory and computer science. * The challenge for students is often determining which elements are the ‘pigeons’ and which are the ‘pigeonholes’ in each problem.   **The naive form of the pigeonhole principle**   * If more than n pigeons are placed into n pigeonholes, then at least one pigeonhole must contain more than one pigeon.   **Examples**   * If I have three boxes how many balls must I have to ensure that one box contains two balls once all the balls are placed into the boxes? * Show that, given a set of n positive integers, there exists a non-empty subset whose sum is divisible by n.   **The basic pigeonhole principle**   * If there are 𝑛 pigeonholes and 𝑛 + 1 pigeons to go into them, then at least one pigeonhole must hold 2 or more pigeons.   **Examples**   * What is the minimum number of students in a school in order for us to guarantee that at least two students will have the same initials? * Given 5 points randomly placed inside of a 2x2 box, prove that there must exist a pair of two points that are within of each other. Note: This [desmos activity](https://www.desmos.com/calculator/xgpmquullz) can be used to explore this problem.   **The generalised pigeonhole principle**   * The pigeonhole principle can be generalised to: If 𝑛 pigeons are sitting in 𝑘 pigeonholes, where 𝑛 > 𝑘, then there is at least one pigeonhole with at least pigeons in it.  **Note:** must always be rounded up to the nearest integer. This is called a ceiling function and can be denoted as .   **Examples**   * There are 50 baskets of apples. Each basket contains no more than 24 apples. Show that there are at least 3 baskets containing the same number of apples. * Show that among any 4 numbers one can find 2 numbers so that their difference is divisible by 3   **Prove the pigeonhole principle**   * The pigeonhole principle can be proved by contradiction or by mathematical induction. It is suggested that proof by contradiction is most appropriate at this stage. There is opportunity to revisit the proof by induction in the Extension 2 topic Further Proof by Mathematical Induction. * **Proof by contradiction:**  Assume there is no pigeonhole with at least pigeons. Therefore, every hole has pigeons Total number of pigeons would be pigeons,  or pigeons.  This says that the total number of pigeons is strictly **less than** but we know that there are exactly pigeons. This means that our assumption was incorrect and that the generalised PHP is true. |  |  |
| Using Pascal’s triangle to expand binomial product  (1 lesson) | **A1.2: The binomial expansion and Pascal’s triangle**   * expand for small positive integers (ACMMM046)   + note the pattern formed by the coefficients of in the expansion of and recognise links to Pascal’s triangle | **Using Pascal’s triangle to expand binomial products**   * Pascal’s triangle can be used to expand for particular values of .   **Example**   * Expand |  |  |
| Investigating the expansion of binomial products  (2 lessons) | * expand for small positive integers (ACMMM046)   + recognise the numbers (also denoted) as binomial coefficients (ACMMM047) | **Investigating the expansion of binomial products**   * Expand for general n by considering the number of ways of obtaining a term of the form   Example   * What is the coefficient of in the expansion of ? * From the 3 brackets we choose one y which can be done in ways. So the coefficient of is . * Generalising this to find the coefficient of in the expansion of , we could select from the first brackets and from the next brackets. Now the number of ways of arranging these (because some are the same) is . Hence the coefficient of is . * The binomial theorem * Students could investigate a further [proof](http://www.amsi.org.au/ESA_Senior_Years/SeniorTopic1/1_md/SeniorTopic1c.html#content_3) of the binomial theorem. Proof by mathematical induction of the binomial theorem could be revisited in the Extension 2 topic Further Proof by Mathematical Induction.   **Example**   * Expand   **Applications of the binomial theorem**   * The general term in the expansion of can be written as * Use a general term of the expansion of to find a specific term in the expansion   Examples   * Find coefficient of in expansion * Find term in in expansion of * Find values of a and n if ... are the first three terms of * **Note:** Greatest coefficient is no longer is not required. It may be explored as possible extension. * **Extension:** Use the general term of the expansion of to find the greatest term in an expansion. **e.g.** Find the greatest term in the expansion of |  |  |
| Understanding the properties of Pascal’s triangle  (1 lesson) | * derive and use simple identities associated with Pascal’s triangle (ACMSM009)   + establish combinatorial proofs of the Pascal’s triangle relations   ;  for ;  and | **Understanding the properties of Pascal’s triangle**   * Derive and use simple identities associated with Pascal’s triangle. * Students should [observe patterns](http://www.amsi.org.au/ESA_Senior_Years/SeniorTopic1/1_md/SeniorTopic1c.html#content_4) in Pascal’s triangle and apply them to different situations. * Students should formalise observations to establish the simple identities:;  for ; and   **Examples**   * Find the value of n if * Explore [Further patterns in Pascal's Triangle](https://www.parabola.unsw.edu.au/2010-2019/volume-52-2016/issue-2/article/pascal%E2%80%99s-many-secrets) |  |  |

Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All ICT, literacy, numeracy and group activities should be recorded in the ‘comments, feedback, additional resources used’ section.