 Year 12 Mathematics Advanced

| MA-C4 Integral calculus | Unit duration |
| --- | --- |
| The topic Calculus involves the study of how things change and provides a framework for developing quantitative models of change and deducing their consequences. It involves the development of two key aspects of calculus, namely differentiation and integration. The study of calculus is important in developing students’ capacity to operate with and model situations involving change, using algebraic and graphical techniques to describe and solve problems and to predict outcomes in fields such as biomathematics, economics, engineering and the construction industry. | 5 weeks |

| Subtopic focus | Outcomes |
| --- | --- |
| The principal focus of this subtopic is to introduce the anti-derivative or indefinite integral and to develop and apply methods for finding the area under a curve, including the Trapezoidal rule and the definite integral, for a range of functions in a variety of contexts. Students develop their understanding of how integral calculus relates to area under curves and a further understanding of the interconnectedness of topics from across the syllabus. Geometrical representation assists in understanding the development of this topic, but careful sequencing of the ideas is required so that students can see that integration has many applications, not only in mathematics but also in other fields such as the sciences and engineering. | A student:* applies calculus techniques to model and solve problems MA12-3
* applies the concepts and techniques of indefinite and definite integrals in the solution of problems MA12-7
* chooses and uses appropriate technology effectively in a range of contexts, models and applies critical thinking to recognise appropriate times for such use MA12-9
* constructs arguments to prove and justify results and provides reasoning to support conclusions which are appropriate to the context MA12-10
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| Prerequisite knowledge | Assessment strategies |
| --- | --- |
| The material in this topic builds on content from MA-C1 Introduction to calculus and MA-C2 Differential calculus. | Formative assessment: Students will investigate anti-derivatives and integration by considering the reverse of the differentiation process to establish the formal process or rule for each type of question. Challenge students to verbalise what they are doing, using generalisations, at each stage of the process. They should draw their responses as a chain of events. Students should be given the opportunity to demonstrate their understanding of the connections between the approximation methods and the calculus methods of integration. |

All outcomes referred to in this unit come from [Mathematics Advanced](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-advanced-2017) Syllabus
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Glossary of terms

| Term | Description |
| --- | --- |
| anti-derivative | An anti-derivative, primitive or indefinite integral of a function is a function whose derivative is , ie The process of finding the anti-derivative is called integration. |
| even function  | Algebraically, a function is even if , for all values of in the domain.An even function has line symmetry about the -axis. |
| odd function  | Algebraically, a function is odd if , for all values of in the domain.An odd function has point symmetry about the origin. |

| Lesson sequence | ContentStudents learn to: | Suggested teaching strategies and resources  | Date and initial | Comments, feedback, additional resources used |
| --- | --- | --- | --- | --- |
| Introducing the anti-derivative(1 lesson) | **C4.1: The anti-derivative*** define anti-differentiation as the reverse of differentiation and use the notation for anti-derivatives or indefinite integrals (ACMMM114, ACMMM115)
* recognise that any two anti-derivatives of differ by a constant
 | **Introducing the anti-derivative*** Introduce the anti-derivative as the result of reversing the process of differentiation. It is an expression of , generally, and is shown as , where.
* The anti-derivation is also known as the indefinite integral or a primitive, and the process of reversing differentiation is known as Integration. The process of integration of a function is notated by and is read as “the integration of with respect to ”.**Resource:** antiderivative-matching-activity.DOCX
 |  |  |
| Determining simple indefinite integrals(1 lesson) | * establish and use the formula , for (ACMMM116)
* recognise and use linearity of anti-differentiation (ACMMM119)
 | **Determining simple indefinite integrals*** Lead students to the rule for integrating polynomial terms, by facilitating examples of the form . A key question may be “Does ?” Discuss that may be a solution but we do not have enough information to tell, as the constant term would have been eliminated during differentiation. To acknowledge the constant in the integral, usually a or is added to the expression,

**i.e.** * Students need to cement their understanding of this concept by being challenged with questions of the form

, , , and **Linearity of integrals*** Establish the following scaling and distributive properties of integrals

and* Students need to apply these properties to solve integrals of the form
 |  |  |
| Establishing the reverse chain rule(2 lessons) | * establish and use the formula where (the reverse chain rule)
* determine indefinite integrals of the form (ACMMM120)
 | **Establishing the reverse chain rule*** Students need to review the chain rule using polynomial expressions, i.e. .
* Challenge students to verbalise what they are doing, using generalisations, at each stage of the process. Draw their responses as a chain of events.
* Highlight the existence of the derivative as a factor of the result of the chain rule for .
* Challenge students to reverse the process by providing them with results from the chain rule and ask them to complete original derivative statement, e.g.
* Consider the result for . How does the result above help us identify the answer? And how does this result help us answer
* By using this result, lead students to the generalised solution to

**Resource**: reverse-chain-rule-matching-activity.docx**Applying the reverse chain rule to functions with a linear expression of x*** Students need to build on the concept of the reverse chain rule for integrals not in the form
* The reverse chain rule can also be applied to integrals in the form , where the expression of is linear. The reverse chain rule cannot be applied to any other expressions of , i.e. the reverse chain rule cannot be applied to as the expression of is not linear.
* Challenge students to develop a solution to an integral of the form
* Lead students to develop the relationship

, where is the primitive of . |  |  |
| Applying standard integrals(2 lessons) | * establish and use the formulae for the anti-derivatives of , and
* establish and use the formulae and
 | **Review standard derivatives*** Review the standard derivatives established in MA-C2 Differential calculus
* Refer to the 2020 HSC reference sheet produced by NESA, which can be found under the Assessment and examination materials on the [NESA Mathematics Advanced syllabus page](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-advanced-2017)

**Applying standard integrals*** Students need to build on their understanding of the reverse chain rule and apply it to a variety of functions
* Lead students to utilise the results on the reference sheet and their understanding of the reverse chain rule to establish
 |  |  |
| Integrals resulting in natural logarithms(1 lesson) | * establish and use the formulae and for , respectively
 | **Integrals resulting in natural logarithms*** Students need to revisit the results for differentiating and in MA-C2 Differential calculus and use the results in reverse to establish the integral results

 where  whereas is undefined* Students are able to solve questions of the form
 |  |  |
| Integrating exponentials of any base(1 lesson) | * establish and use the formulae
 | **Integrating exponentials of any base*** Lead students to establish the formula  by using the following techniques:

Establish and therefore  |  |  |
| Sketching anti-derivatives with slope fields(1 lesson) | * examine families of anti-derivatives of a given function graphically
 | **Sketching anti-derivatives with slope fields*** Introduce students to slope fields using the sketching anti-derivatives activity. A slope field shows the family of curves for the anti-derivative.
* Key questions: “Why is there more than one possible curve?” and “Why are they vertical translations of each other?”
* It is important to establish that integrating alone is not enough to identify the anti-derivative, as there is not enough information to identify the constant term. This can only be established if one point on the curve is known.
* It is important to note that Mathematics Extension 1 students will experience slope fields for expressions that may involve and values. The curves produced in this instance are not vertical translations of each other. **Resource**: sketching-anti-derivative-activity.DOCX
 |  |  |
| Integrating to determine a function(1 lesson) | * determine , given and an initial condition in a range of practical and abstract applications including coordinate geometry, business and science
 | **Integrating to determine a function*** Staff need to build on the key idea that integrating alone is not enough information to determine an anti-derivative as the constant term is unknown.
* The constant term can be determined by substituting in an initial condition , which represents a point on the slope field.

For example, If find given .Step 1: Integrate, Step 2: Substitute in the values of the initial condition to find the constant termStep 3: Determine the anti-derivative function |  |  |
| The area under a curve and its approximations (1 lesson) | **C4.2: Areas and the definite integral*** know that ‘the area under a curve’ refers to the area between a function and the -axis, bounded by two values of the independent variableand interpret the area under a curve in a variety of contexts **AAM**
* determine the approximate area under a curve using a variety of shapes including squares, rectangles (inner and outer rectangles), triangles or trapezia
	+ consider functions which cannot be integrated in the scope of this syllabus, for example , and explore the effect of increasing the number of shapes used
 | **The area under a curve and its approximations** * Introduce area under a curve using a practical context:
	+ Finding the distance travelled from a velocity time graph.
	+ Finding the amount of water wasted from a leaking tap. (Graph water mL/min)
* Define the area under a curve as the area between a function and the x-axis, bounded by two values of the independent variable.
* Explore various methods of approximating areas for curves that cannot be integrated in the scope of this course. Examine the effect of increasing the number of shapes used to approximate the area using the resources below.
* **Resources:** approximating-areas.DOCX, approximating-areas.XLSX
 |  |  |
| Trapezoidal Rule (1 lesson) | * use the notation of the definite integral for the area under the curve from to if
* use the Trapezoidal rule to estimate areas under curves **AAM**
	+ use geometric arguments (rather than substitution into a given formula) to approximate a definite integral of the form , where , on the interval , by dividing the area into a given number of trapezia with equal widths
	+ demonstrate understanding of the formula:

 where and and the values of are found by dividing the interval into equal sub-intervals  | **Trapezoidal Rule** * Define the definite integral for the area under the curve from to if
* Approximate a definite integral by dividing the area into a given number of trapezia with equal widths, calculating the areas of individual trapezia, then the total area.
* Lead the discovery of the trapezoidal rule.

Suggested method: Find an expression for the area of a number of trapezia, repeat for another number of trapezia, then generalise for n trapezia.**NESA exemplar questions*** The following table shows the velocity (in metres per second) of a moving object evaluated at 10-second intervals. Use the trapezoidal rule to obtain an estimate of the distance travelled by the object over the time interval .

Table of values for velocity against time.Discuss other methods for obtaining the estimate.* An object is moving on the -axis. The graph shows the velocity, , of the object as a function of time .

The coordinates of the points shown on the graph are and . The velocity is constant for . page 135 adv* + Use the trapezoidal rule to estimate the distance travelled between and (noting that distance is given on a velocity-time graph by the area under the graph).
	+ The object is initially at the origin. When is the displacement of the object decreasing?
	+ Estimate the time at which the object returns to the origin. Justify your answer.
	+ Sketch the displacement as a function of time.
 |  |  |
| Integrals using the area of basic shapes (1 lesson) | * use geometric ideas to find the definite integral where is positive throughout an interval and the shape of allows such calculations, for example when is a straight line in the interval or is a semicircle in the interval **AAM**
 | **Integrals using the area of basic shapes*** Students apply the area of basic shapes to find the definite integral where is positive throughout an interval .
* Examples in desmos:
	+ [Triangle](https://www.desmos.com/calculator/c4sukxq9lm)
	+ [Trapezium](https://www.desmos.com/calculator/xwabx3jxin)
	+ [Circle](https://www.desmos.com/calculator/9g7okfagav)
	+ [Semi-circle](https://www.desmos.com/calculator/80tr2pv5a1)

**NESA exemplar questions**Graphs A and B shown below represent the functions  and respectively. Graph AGraph B * Evaluate the integral
* Use the formula for the area of a circle to find
 |  |  |
| Signed areas (1 lesson) | * understand the relationship of position to signed areas, namely that the signed area above the horizontal axis is positive and the signed area below the horizontal axis is negative
* using technology or otherwise, investigate the link between the anti-derivative and the area under a curve
* interpret as a sum of signed areas (ACMMM127)
* understand the concept of the signed area function (ACMMM129)
 | **Signed areas*** Students investigate definite integrals to develop an understanding of signed areas.

Students could* + Evaluate using an integral calculator such as [wolfram alpha](https://www.wolframalpha.com/calculators/integral-calculator/).

Enter: integrate 2x-6 from 1 to 7* + Examine the individual areas using [desmos](https://www.desmos.com/calculator/be5ne9vwi8). Enter the functions and bounds
	+ Interpret the result: as the sum of signed areas, area above the horizontal axis is positive and the area below the horizontal axis is negative.
	+ Note: The area between and the x-axis bounded by and 7 is , the corresponding definite integral equals
* Define the area under a curve as:
	+ where f(x) is positive for .
	+ where f(x) is negative for .
 |  |  |
| Definite integrals (1 lesson) | * use the formula where is the anti-derivative of to calculate definite integrals (ACMMM131) **AAM**
	+ understand and use the Fundamental Theorem of Calculus, and illustrate its proof geometrically (ACMMM130)
	+ calculate total change by integrating instantaneous rate of change
 | **Definite Integrals:*** [Demonstration of increasing the number of shapes as a link to integration (Geogebra)](https://www.geogebra.org/m/nmw6Dhdk)
* Prove using an understanding of the fundamental theorem of calculus.

NESA’s Mathematics Advanced Year 12 [topic guidance](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-advanced-2017) for calculus contains two approaches.* Use the formula where is the anti-derivative of to calculate definite integrals.
* Calculate total change by integrating instantaneous rate of change. Examples:
	+ Given a function of velocity, . Integrate the instantaneous rate of change (velocity) to find total displacement for a given time period..
	+ Given a function of acceleration. Integrate the instantaneous rate of change to find total change in velocity.
 |  |  |
| Area under a curve(1 lesson) | * calculate the area under a curve (ACMMM132)
	+ use symmetry properties of even and odd functions to simplify calculations of area
	+ recognise and use the additivity and linearity of definite integrals (ACMMM128)
 | **Area under a curve:*** Apply definite integrals to calculate the area under curves including:
	+ Area bounded by the curve and the x-axis
	+ Area between set bounds (e.g. and )
	+ Area between set bounds where f(x) changes sign.

Teacher to model dissecting the definite integral where f(x) changes sign.The teacher can use a [desmos resource](https://www.desmos.com/calculator/be5ne9vwi8) to construct graphs for demonstrations of finding the area under a curve. * Properties useful for calculations of area:
	+ Even and odd functions.
	+ Additivity of definite integrals:

[Visual representation of the sum of additivity](https://www.geogebra.org/m/s2asvWnh)* + Linearity of definite integrals:

**NESA exemplar questions*** Find the area bounded by the graph of , the -axis, and the lines and .
* Show that .
* Explain why this is not representative of the area bounded by the graph of , the-axis, and the lines and .
 |  |  |
| Area between two curves (1 lesson) | * calculate areas between curves determined by any functions within the scope of this syllabus (ACMMM134) **AAM**
 | **Area between two curves:*** Apply definite integrals to calculate the area between two curves including:
	+ Area between two curves over set bounds

[Visual representation of area between 2 curves](https://www.geogebra.org/m/MpF9wtfA)* + Area between two curves (bounds defined by the points of intersection)

[Visual representation of subtracting the areas.](https://www.geogebra.org/m/YCt3bvBE)* + Area between two curves over set bounds where f(x) and g(x) intersect. Teacher to model dissecting the definite integral where f(x) and g(x) intersect.

[Visual representation of area between 2 curves](https://www.geogebra.org/m/MpF9wtfA)Note: equations need to be adjusted to examine when curves intersect.The teacher can also use a [desmos resource](https://www.desmos.com/calculator/x3nts4fqb9) to construct graphs for demonstrations of finding the area between curves. **NESA exemplar questions*** Find the area bounded by the line and the curve .
* Sketch the region bounded by the curve and the lines Evaluate the area of this region.
 |  |  |
| Problem solving (1 lesson) | * integrate functions and find indefinite or definite integrals and apply this technique to solving practical problems **AAM**
 | **Problem solving*** Trapezoidal rule: See NESA exemplar questions in this unit.
* Economics: Consumer and producer surplus
* Average value of a function
* Work
* Braking Distance

**Resource:** integral-calculus-applications.DOCX |  |  |

Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All ICT, literacy, numeracy and group activities should be recorded in the ‘Comments, feedback, additional resources used’ section.