 Year 11 mathematics advanced

| MA-C1 Introduction to differentiation | Unit duration |
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| The topic Calculus is concerned with how quantities change and provides a framework for developing quantitative models of change and deducing their consequences. The topic involves the development of the basic concepts upon which differential calculus is built, namely the connection between the gradient of the tangent to a curve and the instantaneous rate of change of a function, rates of change and derivatives of functions and the manipulative skills necessary for the effective use of differential calculus. The study of calculus is important in developing students’ ability to solve problems involving algebraic and graphical representations of functions and rates of change of a function with relevance to all quantitative fields of study including physics, chemistry, medicine, engineering, computing, statistics, business, finance, economics and the construction industry. | 4 weeks |

| Subtopic focus | Outcomes |
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| The principal focus of this subtopic is for students to develop an understanding of the concept of a derivative as a function that defines the rate of change of a given function. This concept is reinforced numerically, by calculating difference quotients, geometrically, as gradients of secants and tangents, and algebraically. The derivatives of power functions are found and used to solve simple problems, including calculating gradients and equations of tangents and normals. Students develop an understanding of derivatives as representations of rates of change. This process is of fundamental importance in Mathematics and has applications in all quantitative fields of study including physics, chemistry, medicine, engineering, computing, statistics, business, finance and economics. | A student:* uses algebraic and graphical techniques to solve, and where appropriate, compare alternative solutions to problems MA11-1
* interprets the meaning of the derivative, determines the derivative of functions and applies these to solve simple practical problems MA11-5
* uses appropriate technology to investigate, organise, model and interpret information in a range of contexts MA11-8
* provides reasoning to support conclusions which are appropriate to the context MA11-9
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| Prerequisite knowledge | Assessment strategies |
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| The material in this topic builds on content from MA-F1 and Stage 5 Surds and Index Laws. | Teachers should consider using the various online resources within this unit as formative assessments. These resources require students to make connections between functions and their derivatives in real-world applications. |

All outcomes referred to in this unit come from [Mathematics Advanced](http://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-advanced-2017) Syllabus
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Glossary of terms

| **Term** | **Description** |
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| acceleration | In formal terms, acceleration is the rate of change of velocity over time. In more informal terms, it represents to increase or decrease in speed. |
| angle of inclination **** | The angle of inclination of a straight line is the angle the line makes with the positive x-axis. |
| derivative | The derivative represents the rate of change of a function, $y$, with respect to $x$ and is represented as $\frac{dy}{dx}$. |
| displacement | Displacement is a vector quantity that is associated with the scalar quantity of distance. The magnitude of displacement can be considered as the distance between initial point and the final point of the vector. The direction from the initial point to the final point indicates the direction of the vector. |
| instantaneous rate of change **** | The instantaneous rate of change is the rate of change at a particular moment. For a differentiable function, the instantaneous rate of change at a point is the same as the gradient of the tangent to the curve at that point. This is defined to be the value of the derivative at that particular point. |
| limit**** | The limit of a function at a point $a$, if it exists, is the value the function approaches as the independent variable approaches $a$.The notation used is: $\lim\_{x\to a}f\left(x\right)=L$This is read as ‘the limit of $f(x)$ as $x$ approaches $a$ is $L$’. |
| normal to a curve **** | In calculus, the normal to a curve at a given point P is the straight line that is perpendicular to the tangent to the curve at that point. |
| polynomial | A polynomial is an expression of the form $a\_{n}x^{n}+a\_{n-1}x^{n-1}+ . . . +a\_{2}x^{2}+a\_{1}x+a\_{0}$, where $n$ is a non-negative integer. |
| stationary point **** | A stationary point on the graph $y=f(x)$ of a differentiable function is a point where $f'(x)=0$.A stationary point could be classified as a local or global maximum or minimum or a horizontal point of inflection. |
| tangent | The tangent to a curve at a given point P can be described intuitively as the straight line that ‘just touches’ the curve at that point. At P the curve has ‘the same direction’ as the tangent. In this sense it is the best straight-line approximation to the curve at point P. |

| Lesson sequence | ContentStudents learn to: | Suggested teaching strategies and resources  | Date and initial | Comments, feedback, additional resources used |
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| Investigation of continuous and discontinuous functions (1 lesson) | **C1.1: Gradients of tangents*** distinguish between continuous and discontinuous functions, identifying key elements which distinguish each type of function
	+ sketch graphs of functions that are continuous and compare them with graphs of functions that have discontinuities
	+ describe continuity informally, and identify continuous functions from their graphs
 | **Investigation into continuity*** Review the concept of function
* [Polygraph](https://teacher.desmos.com/polygraph/custom/560aa8e69e65da561507a64d): Continuity game - students try to guess each other’s graphs. Great way to introduce vocabulary of continuous, discontinuous, defined, undefined, hole, asymptote, and limit
* Exit slip assessing students’ understanding of continuity and discontinuity. **Resource:** [exit-slips-contiuous-and-discontinuous-functions.PDF](https://drive.google.com/file/d/15I-7bn4C5kjrpNAPjJHtKBLEt0jdfbhc/view?usp=sharing)
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| Describing the gradient of a curve as the gradient of the tangent at that point(1 lesson) | * describe the gradient of a secant drawn through two nearby points on the graph of a continuous function as an approximation of the gradient of the tangent to the graph at those points, which improves in accuracy as the distance between the two points decreases
* examine and use the relationship between the angle of inclination of a line or tangent, $θ,$ with the positive $x$-axis, and the gradient, $m$, of that line or tangent, and establish that $\tan(θ)=m$ **AAM**
 | **Describing the gradient of a curve*** Use various [Scootle activities](http://www.scootle.edu.au/ec/viewing/R10711/index.html) that lead from rates of change to differential calculus. During these activities students are introduced to the gradient of secants for linear, quadratic, cubic and quartic equations by using distance-time graphs and reducing time intervals to highlight the relationship between average and instantaneous velocity.
* Investigate the connection between the gradient and tangent ratio using DESMOS. **Resource:** the-gradient-and-tangent-ratio.DOCX
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| Describing the behaviour of a function(1 lesson) | **C1.2: Difference quotients*** describe the behaviour of a function and its tangent at a point, using language including increasing, decreasing, constant, stationary, increasing at an increasing rate **AAM** Critical and creative thinking icon Literacy icon
* interpret and use the difference quotient $\frac{f(x+h)-f(x)}{h}$ as the average rate of change of $f(x)$ or the gradient of a chord or secant of the graph $y=f(x)$ Critical and creative thinking icon Literacy icon
* interpret the meaning of the gradient of a function in a variety of contexts, for example on distance–time or velocity–time graphs Critical and creative thinking icon Literacy icon
 | **Difference quotients*** The collection of [Scootle activities](http://www.scootle.edu.au/ec/viewing/R10711/index.html) also introduces the derivative using first principles for all power functions. It enables students to investigate the relationships between power functions and their derivatives.
* Use the resource [From Secant to Tangent](https://www.geogebra.org/m/PXRTrU47) to investigate gradient of a secant as the points come closer together.
* Students to explore the algebraic development of difference quotient, connecting the algebra to the relationships they have explored through the Scootle and Geogebra resources.
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| Differentiation from first principles(2 lessons) | **C1.3: The derivative function and its graph*** examine the behaviour of the difference quotient $\frac{f(x+h)-f(x)}{h}$ as $h\rightarrow 0$ as an informal introduction to the concept of a limit (ACMMM081)
* interpret the derivative as the gradient of the tangent to the graph of $y=f(x)$ at a point $x$ (ACMMM085)
* estimate numerically the value of the derivative at a point, for simple power functions (ACMMM086) Critical and creative thinking icon  Information and communication technology capability icon
* define the derivative $f^{'}\left(x\right)$ from first principles, as $\lim\_{h\to 0}\frac{f(x+h)-f(x)}{h}$ and use the notation for the derivative: $\frac{dy}{dx}=f^{'}\left(x\right)=y'$, where $y=f(x)$
* use first principles to find the derivative of simple polynomials, up to and including degree 3
* understand the concept of the derivative as a function (ACMMM089)
 | **Formalising differentiation from first principles*** Use the various resources that have been explored in the previous lessons to introduce the concept of a limit to determine a derivative by first principles
* [Excel investigation of gradient of a curve.](https://www.stem.org.uk/resources/elibrary/resource/35716/gradient) This resource from stem.org.uk requires students or staff to register for free. The resource allows students to investigate the gradient of curves by examining the gradients of the chords and tangents at a specific point. The resource provides opportunities to establish a rule for differentiating powers and opportunities to apply it.
* Students to algebraically find the derivative of simple polynomials up to and including degree 3 by first principles.
* Use the resources [Investigating the Gradient Function](https://www.geogebra.org/m/PXRTrU47) and [The Relationship between f anf f’](https://www.geogebra.org/m/PXRTrU47) to support students’ conceptual understanding of the derivative as function.
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| Differentiating a variety of functions(1 lesson) | * use the formula $\frac{d}{dx}(x^{n})=nx^{n-1}$ for all real values of $n$ Critical and creative thinking icon
* differentiate a constant multiple of a function and the sum or difference of two functions Critical and creative thinking icon  Information and communication technology capability icon
 | **Applying rules for differentiating powers*** Practise differentiating a broad range of different functions
* [Differentiation Loop Cards from tes.com](https://www.tes.com/en-au/teaching-resource/calculus-differentiation-loop-cards-11005720). Students or staff will need to register for free to access this resource.
* Staff may like to access the following activities and resources
* A [webpage introducing differentiation](https://nrich.maths.org/4722) from nrich.maths.org.
* A [workbook introducing differentiation](https://www.amsi.org.au/ESA_Senior_Years/PDF/IntroDiffCall3b.pdf) from amsi.org.au.
* Staff may like to access the following activities and resources from tes.com. Staff and students can register by signing in with a Google account, using their @education.nsw.gov.au login or another Google account.
* [Core 1 – Introduction to Differentiation for tes.com](https://www.tes.com/teaching-resource/core-1-introduction-to-differentiation-3003913). This resource includes an introductory presentation with accompanying worksheets.
* [Introduction to Differentiation](https://www.tes.com/teaching-resource/introduction-to-differentiation-11169150). This resource includes a simple presentation with examples.
* [A Level Maths C1: Differentiation Worksheets](https://www.tes.com/teaching-resource/a-level-maths-c1-differentiation-worksheets-6146718). This resources includes activities and worksheets for differentiating power terms.
* Exit slip assessing initial understanding of differentiation. **Resource**: [exit-slips-introductory-differentiation.PDF](https://drive.google.com/file/d/1HVPki79SL_Lk6NNamJwv-R3JkSSNMMlJ/view?usp=sharing)
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| Sketching derivative functions from a given graph (2 lessons) | * sketch the derivative function (or gradient function) for a given graph of a function, without the use of algebraic techniques and in a variety of contexts including motion in a straight line Critical and creative thinking icon  Information and communication technology capability icon Literacy icon
	+ establish that $f^{'}\left(x\right)=0$ at a stationary point, $f^{'}\left(x\right)>0$ when the function is increasing and $f^{'}\left(x\right)<0$ when it is decreasing, to form a framework for sketching the derivative function
	+ identify families of curves with the same derivative function (ACMMM121)
	+ use technology to plot functions and their gradient functions
* interpret and use the derivative at a point as the instantaneous rate of change of a function at that point **AAM**
	+ examine examples of variable rates of change of non-linear functions (ACMMM087)
 | **Sketching derivative functions*** Graphing gradient function from original graph – worksheet and solution
* [Derivative matching activity from stem.org.uk](https://www.stem.org.uk/resources/elibrary/resource/31126/derivative-matching). Students need to match an equation with its graph and its derivative equation with its derivative graph.
* [Matching function and derivative equations from stem.org.uk](https://www.stem.org.uk/system/files/elibrary-resources/legacy_files_migrated/6523-C3.pdf). Students will need to be familiar with function notation and its equivalent derivative notation.
* Use technology to show the relationship between gradient function and original function
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| Applying the product rule(1 lesson) | * understand and use the product, quotient and chain rules to differentiate functions of the form $f\left(x\right)g\left(x\right), \frac{f(x)}{g(x)}$ and $f\left(g\left(x\right)\right)$ where $f(x)$ and $g(x)$ are functions
	+ apply the product rule: If $h(x)=f(x) g(x)$ then $h'(x)=f(x) g'(x) + f'(x) g(x)$, or if $u$ and $v$ are both functions of $x$ then $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$
 | **Applying the product rule*** Students should be encouraged to investigate deriving the product rule for differentiation from first principles. This [webpage for quora.com](https://www.quora.com/How-do-you-prove-the-product-rule-of-differentiation) outlines the simple algebraic technique used prior to taking limits and determining the rule.
* Staff should only concentrate on applying the product rule with power functions, at this stage. Staff should invite discussions regarding suitable methods and alternative approaches. For example, the methods of product rule and expanding brackets should be discussed for derivatives in the form $\frac{d}{dx}\left[x^{2}\left(3x-2\right)\right]$
* Staff may like to use [this Introduction to the Product Rule resource](http://www.mathcentre.ac.uk/resources/uploaded/mc-ty-product-2009-1.pdf) from mathcentre.ac.uk.
* Staff and students may like to use [this derivative calculator](https://www.derivative-calculator.net/) to check answers and to show steps for differentiation
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| Applying the quotient rule(1 lesson) | * understand and use the product, quotient and chain rules to differentiate functions of the form $f\left(x\right)g\left(x\right), \frac{f(x)}{g(x)}$ and $f\left(g\left(x\right)\right)$ where $f(x)$ and $g(x)$ are functions
	+ apply the quotient rule: If $h(x)=\frac{f(x)}{g(x)}$ then $h'(x)=\frac{g(x)f^{'}(x)-f(x)g^{'}(x)}{g(x)^{2}}$, or if $u$ and $v$ are both functions of $x$ then $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^{2}}$
 | **Applying the quotient rule*** Students should be encouraged to investigate deriving the quotient rule for differentiation by expanding on the idea of the product rule. This [Khan Academy video](https://www.khanacademy.org/math/ap-calculus-ab/ab-differentiation-2-new/ab-diff-2-optional/v/quotient-rule-from-product-rule) illustrates the technique used.
* Students may like to try this [product and quotient rule quiz](http://www.maths.usyd.edu.au/u/UG/JM/MATH1111/Quizzes/quiz14.html) from maths.usyd.edu.au.
* Staff and students may like to use [this derivative calculator](https://www.derivative-calculator.net/) to check answers and to show steps for differentiation
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| Applying the chain rule(1 lesson) | * understand and use the product, quotient and chain rules to differentiate functions of the form $f\left(x\right)g\left(x\right), \frac{f(x)}{g(x)}$ and $f\left(g\left(x\right)\right)$ where $f(x)$ and $g(x)$ are functions
	+ apply the chain rule: If $h(x)=f(g(x))$ then $h'(x)=f'(g(x)) g'(x)$, or if$ y$ is a function of $u$ and $u$ is a function of $x$ then $\frac{dy}{dx} = \frac{dy}{du} × \frac{du}{dx}$
 | **Applying the chain rule*** Students should investigate the chain rule by using the product rule to explore simple cases starting with

$$\frac{d}{dx}\left[\left(f\left(x\right)\right)^{2}\right]$$$$=\frac{d}{dx}\left[f\left(x\right).f\left(x\right)\right]$$$$=f^{'}\left(x\right)f\left(x\right)+f\left(x\right).f^{'}\left(x\right)$$$$=2f\left(x\right)f'(x)$$And expanding onto$$\frac{d}{dx}\left[\left(f\left(x\right)\right)^{3}\right]$$$$=\frac{d}{dx}\left[f\left(x\right).\left(f\left(x\right)\right)^{2}\right]$$$$=f^{'}\left(x\right).\left(f\left(x\right)\right)^{2}+f\left(x\right).2f(x)f^{'}\left(x\right)$$$$=3\left(f\left(x\right)\right)^{2}f'(x)$$Students are encouraged to describe any patterns or relationships to develop an initial rule for using the chain rule with powers. Staff need to establish this as an introduction into the chain rule and there is a formal definition for the chain rule.* Students may like to try this [chain rule quiz from maths.usyd.edu.au](http://www.maths.usyd.edu.au/u/UG/JM/MATH1111/Quizzes/quiz33.html).
* Staff and students may like to use [this derivative calculator](https://www.derivative-calculator.net/) to check answers and to show steps for differentiation
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| Real life applications of power functions and derivatives(1 lesson) | * calculate derivatives of power functions to solve problems, including finding an instantaneous rate of change of a function in both real life and abstract situations **AAM**
 | **Real life applications*** Students are to be exposed to a number of real life and abstract problems to solve. Many of these contexts should have already been explored in this unit but could include finding instantaneous rates of change scenarios in the contexts of changing volumes, heights, distances, populations, masses or areas.
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| Determining the equations of tangents and normal(1 lesson) | * use the derivative in a variety of contexts, including finding the equation of a tangent or normal to a graph of a power function at a given point **AAM**
 | **Determining the equations of tangents and normals*** [Excel spreadsheet](https://www.stem.org.uk/resources/elibrary/resource/35720/tangents-and-normals) showing the tangent and normal to different curves
* [Equations of tangents and normal tarsia](https://www.tes.com/en-au/teaching-resource/equations-of-tangents-and-normals-tarsia-6395260)
* Exit slip assessing students’ ability to apply differentiation techniques to find the equation of the tangent or normal. **Resource**: [exit-slips-tangents-and-normals.PDF](https://drive.google.com/file/d/1GzQ4crsKbH_BVCVOTzp7XaC42g3nEVGQ/view?usp=sharing)
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| Finding the velocity and acceleration(1 lesson) | * determine the velocity of a particle given its displacement from a point as a function of time
* determine the acceleration of a particle given its velocity at a point as a function of time
 | **Finding the velocity and acceleration*** If the displacement of a particle, $x$, is defined as a function of time, $x=f\left(t\right)$, lead students to determine
	+ $\frac{dx}{dt}=f^{'}\left(x\right)$ and velocity, $v=\frac{dx}{dt}$.

Therefore $v=f'(x)$.* + $\frac{d^{2}x}{dt^{2}}=f''(x)$ and acceleration $a=\frac{d^{2}x}{dt^{2}}$.

Therefore $a=f''(x)$* [Velocity Time Graphs DESMOS](https://teacher.desmos.com/polygraph/custom/560ad68f7701c303063305f5)
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Reflection and evaluation

Please include feedback about the engagement of the students and the difficulty of the content included in this section. You may also refer to the sequencing of the lessons and the placement of the topic within the scope and sequence. All ICT, literacy, numeracy and group activities should be recorded in the ‘comments, feedback, additional resources used’ section.