****Mathematics Extension 2****

# MEX-P1 The Nature of Proof

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**Disclaimer**

This document is to be used to supplement the support teachers are offering students undertaking HSC Mathematics courses. Questions can be printed off for students individually, with or without solutions, or as an entire booklet. Questions have been sourced from various states across Australia and the source of each question has been referenced. Permission to use these resources was provided in June 2020. Solutions for each of the questions can be found at the end of the document.

**Outcomes**

All outcomes referred to in this booklet are from [Mathematics Extension 2 Syllabus](https://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-extension-2-2017) © 2017 NSW Education Standards Authority (NESA) for and on behalf of the Crown in right of the State of New South Wales.

## Outcomes

**A student:**

* understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts MEX12-1
* chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings MEX12-2
* applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems MEX12-7
* communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument MEX12-8

## Content

### MEX-P1 The Nature of Proof

Students:

* use the formal language of proof, including the terms statement, implication, converse, negation and contrapositive (ACMSM024)
	+ use the symbols for implication , equivalence and equality , demonstrating a clear understanding of the difference between them (ACMSM026)
	+ use the phrases ‘for all’ , ‘if and only if’ and ‘there exists’ (ACMSM027)
	+ understand that a statement is equivalent to its contrapositive but that the converse of a true statement may not be true
* prove simple results involving numbers (ACMSM061)
* use proof by contradiction including proving the irrationality for numbers such as and (ACMSM025, ACMSM063)
* use examples and counter-examples (ACMSM028)
* prove results involving inequalities. For example:
	+ prove inequalities by using the definition of for real and
	+ prove inequalities by using the property that squares of real numbers are non-negative
	+ prove and use the triangle inequality and interpret the inequality geometrically
	+ establish and use the relationship between the arithmetic mean and geometric mean for two non-negative numbers
* prove further results involving inequalities by logical use of previously obtained inequalities

## Supplementary resources

### Department of Education resources

#### Units of work

* [MEX-P1 The nature of proof](https://education.nsw.gov.au/teaching-and-learning/curriculum/key-learning-areas/mathematics/stage-6/mathematics-extension-2)
* [MEX P1 sample question solutions](https://education.nsw.gov.au/teaching-and-learning/curriculum/key-learning-areas/mathematics/stage-6/mathematics-extension-1#Year10)

### NESA resources

* [Mathematics Extension 2 – Sample examination materials (2020)](https://educationstandards.nsw.edu.au/wps/portal/nesa/11-12/stage-6-learning-areas/stage-6-mathematics/mathematics-extension-2-2017)

### WOOTUBE

* [The Nature of Proof](https://www.youtube.com/playlist?list=PL5KkMZvBpo5DdkpsH2qYqaTHUnaERss7y)

### Jonathan Kim Sing videos

* [Further Proof by Mathematical Induction (Y12 Extension 2)](https://www.youtube.com/playlist?list=PLnPz4TGRkQDtHNlpXNE3e9GSm20i2unxe)

## Examination-style questions

### Sample question 1

**Question 7**

Consider the set of odd numbers {1, 3, 5, 7 …}.

1. Show that the following statement is false by providing a counter-example. **(1.5 marks)**

‘The square of an odd number is always an even number.’

1. Prove that the square of any odd number always produces another odd number.
**(3.5 marks)**

Source:[©QCAA 2018 Specialist mathematics sample assessment instrument: Examination (15%)](https://www.qcaa.qld.edu.au/senior/senior-subjects/mathematics/specialist-mathematics/assessment)

### Sample question 2

**Question 8**

Consider the following statement:

‘If two vectors are perpendicular, then their scalar product is zero’

1. State the contraposition of this statement. **(1 mark)**
2. State whether the contraposition statement is true or false. Briefly justify your decision. **(1 mark)**

Source:[©QCAA 2018 Specialist mathematics sample assessment instrument: Examination (15%)](https://www.qcaa.qld.edu.au/senior/senior-subjects/mathematics/specialist-mathematics/assessment)

### Sample question 3

**Question 9**

The initial part of a proof by contradiction to show that is an irrational number is given below.

Assume that is a rational number.

Then can be written as a ratio of integers such that

Where and only have 1 as a common factor

Complete the remainder of this proof by contradiction. **(4 marks)**

Source:[©QCAA 2018 Specialist mathematics sample assessment instrument: Examination (15%)](https://www.qcaa.qld.edu.au/senior/senior-subjects/mathematics/specialist-mathematics/assessment)

### Sample question 4

**Question 11**

Let *n* be a positive integer.

1. Find a counterexample to show that the following statement is false. **(1 mark)**

 is always a prime number

1. i. Write down the contrapositive of: **(1 mark)**

 is even then is odd

 ii. Use the contrapositive to prove that if is even then is odd.
 **(3 marks)**

Source: [Copyright © Scottish Qualifications Authority 2019 Advanced Higher Question paper](https://www.sqa.org.uk/pastpapers/findpastpaper.htm?subject=Mathematics&level=NAH)

### Sample question 5

**Question 9**

Prove directly that:

1. The sum of any three consecutive integers is divisible by 3. **(2 marks)**
2. Any odd integer can be expressed as the sum of two consecutive integers. (**1 mark)**

Source: [Copyright © Scottish Qualifications Authority 2018 Advanced Higher Question paper](https://www.sqa.org.uk/pastpapers/findpastpaper.htm?subject=Mathematics&level=NAH)

### Sample question 6

**Question 13**

Let be an integer.

Using proof by contrapositive, show that if **(4 marks)**

Source: [Copyright © Scottish Qualifications Authority 2017 Advanced Higher Question paper](https://www.sqa.org.uk/pastpapers/findpastpaper.htm?subject=Mathematics&level=NAH)

### Sample question 7

**Question 10 (4 marks)**

For each of the following statements, decide whether it is true of false.

If true, give a proof; if false, give a counterexample.

1. If a positive integer is prime, then so is
2. If a positive integer has remainder 1 when divided by 3, then also has remainder 1 when divided by 3.

Source: [Copyright © Scottish Qualifications Authority 2016 Advanced Higher Question paper](https://www.sqa.org.uk/pastpapers/findpastpaper.htm?subject=Mathematics&level=NAH)

### Sample question 8

**Question 12**

Prove that the difference between the squares of any two consecutive odd numbers is divisible by 8 **(3 marks)**

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### Sample question 9

**Question 11 (5 marks)**

For each of the following statements, decide whether it is true or false and prove your conclusion.

1. For all natural numbers, if is divisible by 4 then is divisible by 4.
2. The cube of any odd integer plus the square of any even integer is always odd

Source: [Copyright © Scottish Qualifications Authority 2008 Advanced Higher Question paper](http://www.dunblanehighschool.org.uk/maths/course/advanced-higher/advanced-higher-past-papers/)

### Sample question 10

**Question 12 (4 marks)**

Prove by contradiction that if is an irrational number, than is irrational

Source: [Copyright © Scottish Qualifications Authority 2010 Advanced Higher Question paper](http://www.dunblanehighschool.org.uk/maths/course/advanced-higher/advanced-higher-past-papers/)

### Sample question 11

**Question 9 (3 marks)**

Prove by contradiction, that if is irrational then is irrational.

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### Sample question 12

**Question 12**

1. Given that and are positive integers state the negation of the statement: **(1 mark)**

is even or is even.

1. By considering the contrapositive of the following statement: **(3 marks)**

if is even then is even or is even,

prove that the statement is true for all positive integers and

Source: [Copyright © Scottish Qualifications Authority AH Specimen paper](http://www.dunblanehighschool.org.uk/maths/course/advanced-higher/advanced-higher-past-papers/)

### Sample question 13

**Question 4**

Consider the following conjecture:

**If** N is a positive integer that consists of the digit 1 followed by an odd number of 0 digits and then a final digit 1, **then** N is a prime number.

Here are three numbers:

I N = 101 (which is a prime number)

II N=1001 (which equals 7 x 11 x 13)

III N=10001 (which equals 73 x 137)

Which of these provide(s) a counterexample to the conjecture?

1. None of them
2. I only
3. II only
4. III only
5. I and II only
6. I and III only
7. II and III only
8. I, II and III

Source: [(C) Test of Mathematics for University Admission (TMUA) 2019 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 14

**Question 7**

For which of the following statements can the fact that be used to produce a **counterexample**?

1. If , and are positive integers which satisfy the equation , and the three numbers have no common divisor, then two of them are odd and the other even.
2. The equation has no solutions for which are positive integers
3. The equation has no solutions for which are positive integers.
4. If are positive integers which satisfy the equation, then one is the arithmetic mean of the other two.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2019 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 15

**Question 3**

Consider the following statement:

A car journey consists of two parts. In the first part, the average speed is km/h. In the second part, the average speed is km/h. Hence the average speed for the whole journey is km/h.

Which of the following examples of car journeys provide(s) a **counterexample** to the statement?

I In the first part of the journey, the car travels at a constant speed of 50 km/h for 100km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

II In the first part of the journey, the car travels at a constant speed of 50 km/h for one hour. In the second part of the journey, the car travels at a constant speed of 40km/h for one hour.

III In the first part of the journey, the car travels at a constant speed of 50 km/h for 80 km. In the second part of the journey, the car travels at a constant speed of 40 km/h for 100 km.

1. None of them
2. I only
3. II only
4. III only
5. I and II only
6. I and III only
7. II and III only
8. I, II and III

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 16

**Question 6**

Which of the following functions provides a **counterexample** to the statement:

**If** for all real , **then** for all real

1.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 17

**Question 12**

Consider the following statement:

For any positive integer there is a positive integer such that is not prime for any positive integer .

Which one of the following is the negation of this statement?

1. For any positive integer there is a positive integer such that there is a positive integer for which is prime.
2. For any positive integer there is a positive integer such that there is a positive integer for which is not prime.
3. For any positive integer there is a positive integer such that for any positive integer , is not prime.
4. For any positive integer , any positive integer and any positive integer is not prime.
5. There is a positive integer such that for any positive integer there is a positive integer for which is not prime.
6. There is a positive integer such that for any positive integer there is a positive integer for which is prime.
7. There is a positive integer such that for any positive integer and any positive integer , is prime.
8. There is a positive integer and a positive integer for which there is no positive integer for which is prime.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 18

**Question 13**

The following is an attempted proof of the conjecture:

**If**

Suppose , so in particular

Since (I)

It follows that (II)

Therefore (III)

Which factorises to give (IV)

Therefore (V)

Which of the following is the case?

1. The proof is correct
2. The proof is incorrect, and the first error occurs in line (I)
3. The proof is incorrect, and the first error occurs in line (II)
4. The proof is incorrect, and the first error occurs in line (III)
5. The proof is incorrect, and the first error occurs in line (IV)
6. The proof is incorrect, and the first error occurs in line (V)

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 19

**Question 5**

Consider the following three statements:

(1) and are both prime when p is an odd prime

(2) Every prime greater than 5 is of the form for some integer n

(3) No multiple of 7 greater than 7 is a prime.

The result can be used to provide a counterexample to which of the following statements?

1. None of them
2. (1) only
3. (2) only
4. (3) only
5. (1) and (2) only
6. (1) and (3) only
7. (2) and (3) only
8. (1), (2) and (3)

Source: [(C) Test of Mathematics for University Admission (TMUA) 2017 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 20

**Question 9**

Consider the following attempt to prove this true theorem:

Theorem: has no solutions with , and positive integers.

Attempted proof:

Suppose that there are positive integers , and such that

I We have

II Hence

III It follows that and , since

IV Eliminating , we have

V Multiplying out, we have

VI Hence so one of and is zero.

But this is a contradiction to the original assumption that all of a, b and c are positive. It follows that the equation has no solutions.

Comment on this proof by choosing one of the following options:

1. The proof is correct
2. The proof is incorrect and the first mistake occurs in line I.
3. The proof is incorrect and the first mistake occurs in line II.
4. The proof is incorrect and the first mistake occurs in line III.
5. The proof is incorrect and the first mistake occurs in line IV.
6. The proof is incorrect and the first mistake occurs in line V.
7. The proof is incorrect and the first mistake occurs in line VI.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2017 paper 2](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

## Solutions

### Sample question 1

1. is odd and , where is odd. Therefore ‘The square of an odd number is always an even number’ cannot be true
2. Let any odd number , where is an integer.

 is even, therefore is odd and is odd.

Therefore the square of any odd number is always odd.

### Sample question 2

1. If two vectors have a scalar product not equal to zero, then they are not perpendicular.
2. The contraposition is true, as the scalar product is non-zero when the angle between the vectors is not a right-angle as when .

### Sample question 3

 which is a contradiction as 2 is a common factor of and .

### Sample question 4

1. Students must demonstrate a value of and evaluate and communicate that this value is not prime

 E.g. when which is not prime

1. i. write down contrapositive statement
 If is even then is odd

 ii. write down appropriate form for and substitute

 and

 Show is odd
 E.g. which is odd since

 Communicate

 E.g. contrapositive state is true AND therefore original statement is true

Source: [Copyright © Scottish Qualifications Authority 2019 Advanced Higher marking instruction](https://www.sqa.org.uk/pastpapers/findpastpaper.htm?subject=Mathematics&level=NAH)

### Sample question 5

1. which is divisible by 3

Source: [Copyright © Scottish Qualifications Authority 2018 Advanced Higher marking instruction](https://www.sqa.org.uk/pastpapers/findpastpaper.htm?subject=Mathematics&level=NAH)

### Sample question 6

The contrapositive of the original statement is:

If is odd then is odd.

 which is odd

Contrapositive statement is true therefore original statement is true

Source: [Copyright © Scottish Qualifications Authority 2017 Advanced Higher marking instruction](https://www.sqa.org.uk/pastpapers/findpastpaper.htm?subject=Mathematics&level=NAH)

### Sample question 7

1. Choose

and since hence not prime, statement is false

So has remainder 1 when divided by 3 statement is true

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### Sample question 8

Let numbers be

Which is divisible by 8

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### Sample question 9

1. Counter example . So statement is false
2. Let the numbers be

Which is odd

OR

Proving algebraically that either the cube of an odd number is odd or the square of an even number is even.

Odd cubed is odd and even squared is even.

So the sum of them is odd

Source: [Copyright © Scottish Qualifications Authority 2008 Advanced Higher marking instruction](http://www.dunblanehighschool.org.uk/maths/course/advanced-higher/advanced-higher-past-papers/)

### Sample question 10

Assume is rational

And let where are integers

So,

Since and are integers it follows that is rational. This is a contradiction.

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### Sample question 11

 ( natural numbers with no common factor)

 is rational, which is a contradiction, therefore the original statement is true

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### Sample question 12

1. is odd and is odd
2. If and are both odd then is odd

Let where are positive integers.

Then, where is clearly an integer therefore is clearly odd

And so the contrapositive statement is true and it follows that the original statement, ‘if is even then is even or is even’, that is equivalent to the contrapositive, is true.

Source: [Copyright © Scottish Qualifications Authority AH Specimen paper](http://www.dunblanehighschool.org.uk/maths/course/advanced-higher/advanced-higher-past-papers/)

### Sample question 13

A counterexample will have the antecedent (“N is a positive integer that consists of the digit 1 followed by an odd number of 0 digits and then a final digit 1”) true, and the consequent (“N is a prime number”) false.

We can now check each of the three offered values of N:

I The form of N is correct: there is 1 zero, and 1 is odd, so the antecedent is true. However, the consequent is also true, so this is not a counterexample.

II The form of N is not correct, as there are 2 zeros, so the antecedent is false. Hence this is not a counterexample.

III The form of N is correct, as there are 3 zeros, and the consequent is false, as N is not prime. Hence this is a counterexample.

The correct solution is d).

Source: [(C) Test of Mathematics for University Admission (TMUA) 2019 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 14

We work through these in turn:

1. A counterexample here is a case for which the antecedent (the ‘if’ part) is true but the consequent (the ‘then’ part) is false. In this case, taking to match 122 + 162 = 202 does not satisfy the antecedent, as they do have a common divisor of 4. Even if we were to divide all three by 4 to get 32 + 42 = 52, we would still not have a counterexample, as two of them are odd and the other is even. (The statement is actually true, so there are no counterexamples.)
2. A counterexample in this case is an example to show that there is a solution. Since
16 = 42, we can rewrite 122 + 162 = 202 as 44 + 122 = 202, so this does provide a counterexample.
3. Since neither 12 nor 20 is a perfect square, we cannot use this fact to obtain a counterexample. (And that is quite a relief, since Fermat proved that this statement is true! It is the simplest case of Fermat’s Last Theorem.)
4. We note that 122 + 162 = 202 satisfies the antecedent, but as 16 is the arithmetic mean of 12 and 20, this does not provide a counterexample. (The statement is false, and a counterexample is 52 + 122 = 132)

The correct answer is therefore B.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2019 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 15

A counterexample to this statement is an example which shows that it is not true.

So we consider the journeys described, and determine whether or not the average speed for the whole journey is, indeed,

Perhaps the simplest way to do this is to make a table. We know that average speed is distance travelled divided by time, so we will need these for each part of the journey.

Filling in the information given, we end up with the following. We write d­1, d2 and D for the distances (in km) in the first part, second part and total, respectively. Likewise, stands for time (in h), and we use V for the average speed for the whole journey (in km/h).



We can now use u = d1/t1 and so on to fill in the rest of the first and second part of the journey, and then sum the distances and times for the overall journey. We can finally calculate the average speed. This gives:



In every case, so both I and III are counterexamples (as the fractions cannot be simplified, so they cannot equal 45), and the answer is F.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 16

A counterexample to an ‘**if** P **then** Q’ statement is an example where the statement is **false**, that is where P (the antecedent) is **true** but Q (the consequent) is **false**.

We can now look through the given functions to see which of them satisfies these requirements. We give our results as a table and make comments on the working below.



Once we know that for all is **false**, there is no need to go any further, so for A, B and D, we stop at this point.

For C, 3x2 0 for all x, so 3x2 + 1 1 > 0 for all x, and we know that cubics (x3 + x + 1 in this case) always take both positive and negative values, so for all x is **false**.

For E, we don’t know what the derivative of 2x is (or at least, knowledge of it is not required for this test), but we do know that 2x is increasing for all x, so the derivative is positive for all x.

Finally, only C has the antecedent **true** and the consequent **false**, so C provides a counterexample.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 17

The negation of **‘for any** N, P’ is ‘**there exists** N such that **not** P’. So none of options A-D can be a negation of the original statement, but all of E-H still have the potential to be.

In this case, the statement P is ‘there is a positive integer such that is not prime for any positive integer ’; this is a statement of the form ‘**there exists**’, and we wish to negate it. The negation of ‘**there exists** such that ’ is **‘for all** , **not** ’, so we are looking for a statement which continues with ‘for all positive integers ’, and this eliminates option H. (‘For any’ and ‘for all’ are equivalent ways of expressing this.) So we are now looking at options E, F and G.

The statement Q that we wish to negate is ‘ is not prime for any positive integer ’, which we can rewrite with the quantifier (‘for any’) at the start of the statement to give ‘for any positive integer , is not prime’. And the negation of ‘**for all** , R’ is ‘**there exists** such that **not** R’, which in this case is ‘there exists a positive integer such that it is **not** the case that is not prime’, so option G is not correct.

Now ‘**not not** S’ is equivalent to S, so this becomes ‘there exists a positive integer such that is prime’. The only option which matches this is option F, and this is therefore the correct answer.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 18

Commentary: One thing we can do when faced with a question like this it to first try different values of θ to see whether the conjecture appears to be true or false. If it is false, as shown by some counterexample, then substituting this value of θ into every line might indicate where the error arises. If, on the other hand, we cannot find a counterexample, then the conjecture and argument may well be true.

In this case, given that we are dealing with trigonometric functions, we would do well to think about their graphs and try values of θ in different quadrants.

If we sketch (or think about) the graphs of sin, cos and tan, we have the following:



If the conjecture is false, we need a value of where but

We see that but there 1 looks like it might be true. We also have when and there and are both negative, so cannot be true. So if we take we may find the source of the error.

On line (I), we get and so this line looks reasonable.

On line (II), we get which is correct.

On line (III), we get 1 so this is correct.

On line (IV), we get so this line is correct for this value of

On line (V), we have is false. So it appears that the first error is likely to occur here: indeed, we have taken the square root of > 1 and obtained x > 1, which is not necessarily the case, as we could have x < −1

We should now go back and check that the earlier steps are actually valid, as there could have been an earlier error. But with some thinking, we can see that all of them are correct:

(I) This is obtained by multiplying both sides of by (and we cannot have ), and then noting the equality from the definition of given.

(II) This is obtained by doubling the previous inequality and then adding 1.

(III) Here, the identity has been used to rewrite the left hand side.

(IV) This factorisation is correct.

(V) As we saw, an error occurs here.

Hence the answer is F.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2018 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 19

We note from the context of the statements that 91 = 7 × 13 shows that 91 is not prime. It also shows that 91 is a multiple of 7 and a multiple of 13.

We use a counterexample when we are trying to disprove a statement of the form ‘**if** A **then** B’; the counterexample will be a case where A is true but B is false. We can also use counterexamples to disprove statements of the form **‘for all** x, A is true’: a single example of an x for which A is false shows that the statement is false.

We first rewrite each of the statements given into an explicit ‘**if . . . then’** statement or an explicit ‘**for all’** statement.

(With experience, this is not actually necessary; once you have seen enough statements, it becomes clear what a counterexample ‘looks like’. But for the purposes of reaching this point, it can be useful to go through these formal steps.)

(1) The word ‘when’ in this context has the same logical meaning as ‘if’, so the statement can be rewritten as ‘**if** p is an odd prime, **then** 10p2 + 1 and 10p2 − 1 are both prime’.

So if 91 = 7 × 13 is to provide a counterexample, we need an odd prime p with 10p2 + 1 = 91 or 10p2 − 1 = 91, since in that case, ‘10p2 + 1 and 10p2 − 1 are both prime’ will be false. It is clear that p = 3 achieves this, for then 10p2 + 1 = 91.

(2) We can write this statement as an explicit ‘**for all**’ statement, as ‘Every prime. . . ’ suggests this meaning. The statement becomes: ‘**For all** primes p where p > 5, p = 6n + 1 for some integer n’.

So a counterexample would be a prime p with p > 5 but for which p = 6n+1 is false. However, 91 is not prime, so 91 = 7 × 13 does not provide a counterexample. The fact that 91 = 6n + 1 for n = 15 is immaterial, and the fact that the statement is false (as 11 = 6 × 2 − 1) does not help either.

(3) We can again write this as an explicit ‘**for all’** statement; the English is better with the equivalent ‘**for each’**, though: ‘**For each** n which is a multiple of 7 greater than 7, n is not prime.’

If we now look at 91 = 7 × 13, we see that 91 is a multiple of 7 greater than 7. However, 91 is not prime, so 91 is an example of where the statement does hold, rather than being a counterexample. In fact, the given statement is true, so there cannot be any counterexamples, and we need not have looked at 91 = 7 × 13 at all.

So the answer is B.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2017 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)

### Sample question 20

Commentary: This is Fermat’s Last Theorem for the case n = 3; it was first proven by Euler in the 18th century. The attempted proof here, though, is deeply flawed, as we will see.

Line I is fine: this is a simple rearrangement.

Line II is fine: this is a standard algebraic identity which is easy to check.

Line III is problematic. It is certainly true that and

. It is therefore plausible that and , but it is far from necessary. Of course we cannot give an explicit example to show this, as there are no sets of positive integers with . But we can observe that if , we must have , and this possibility has not been considered; alternatively, we could consider a case where a is not prime such as a = 6; then we can write, and we could have and . Either way, it does not necessarily follow that and , so this step is wrong.

The correct answer is D.

Source: [(C) Test of Mathematics for University Admission (TMUA) 2017 paper 2 worked answers](https://www.admissionstesting.org/for-test-takers/test-of-mathematics-for-university-admission/preparation/)